Autocorrelation analysis and statistical consideration for the determination of velocity fluctuations in fusion plasmas

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A new statistical method is proposed and investigated to detect modulation in the poloidal flow velocity via the motion of turbulent eddies. The technique needs a single-point measurement only, and investigates modulation of the autocorrelation function. In order to evaluate the sensitivity of the method an analytical expression is derived for the relative scatter of the autocorrelation function when the fluctuating signal is composed of random events with a given event rate. Detailed formulas are obtained for the case of identical Gaussian pulses. The result of the calculation allows estimation of the scatter of the autocorrelation function due to both event statistical and detector noise. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1909200]

I. INTRODUCTION

Hot plasmas, as a complex system of charged particles, are governed by nonlinear dynamics, hence often they are found in a turbulent state. One example is anomalous transport in fusion plasmas which is generally considered to be a consequence of plasma microturbulence. Recent theories describe a coupled system of turbulence and flows generated by the turbulent Reynolds stress[1] or other secondary nonlinear processes.[2] These flows react on background turbulence by a shear flow decorrelation mechanism,[3] therefore their characterization is essential. Temporally fluctuating and radially localized flows (called zonal flows) have also been seen in recent numerical simulations of fusion plasmas[4] and we have an increasing number of experimental indication to their existence[5–8] as well. However, most of these experimental evidences were collected either in low-temperature devices, where material probes can be immersed into the plasma, or in special diagnostics available only at some machines.

Several of the above experimental results were obtained by measuring changes in the poloidal flow velocity of turbulent structures[9–11] via poloidally resolved fluctuation diagnostics. This paper investigates the possibility of detecting *B*-perpendicular flow modulations through observing temporal variations of the autocorrelation function in a single-point measurement of plasma turbulence. In order to calculate the sensitivity of the method we found it essential to analyze the statistical scatter of the autocorrelation function due to two effects: the finite number of turbulent eddies and/or detector statistical noise. The final result of this derivation is a simple expression readily usable to various measurements. As an application of the method we present the analysis of the dominant error source in beam emission spectroscopy (BES) turbulence measurements on the Wendelstein 7-AS stellarator.[12]

II. EXTRACTION OF INFORMATION ABOUT VELOCITY FLUCTUATIONS FROM CORRELATION FUNCTIONS

Correlation analysis is a useful and well-known tool in turbulence studies of fusion plasmas. Measuring correlation functions (autocorrelation and cross correlation) one can obtain a detailed picture about the space-time structure of turbulence. As we would like to detect fluctuations in the poloidal flow velocity, we need an idea about the motion of the turbulent structures. In experiments, e.g., Doppler reflectometry, one can detect the group velocity of density perturbations in a given poloidal plane, which has the form $v_\perp = v_{E\times B} + v_{\text{turb}}$, where $v_{E\times B}$ is the background drift flow and v_{turb} the intrinsic phase velocity of the vortex modes. Doppler reflectometry results show[13] that the group velocity of propagating density perturbations follows the profile of $E \times B$ velocity measured with passive spectroscopy or with active charge exchange recombination spectroscopy, indicating that the intrinsic phase velocity of the perturbations riding on the background plasma must be small.

Investigating correlation functions we can get information about statistical properties of turbulent structures: the correlation time, the correlation length along a spatial coordinate, and the propagation velocity from cross-correlation function. From a very simple model which deals with poloidally moving structures having Gaussian shape both in space and time, we can calculate the autocorrelation time in a single-point measurement as a function of the eddy lifetime τ_{life} and the velocity dependent propagation time τ_v:

$$\tau_{\text{corr}} = \frac{\tau_{\text{life}}\tau_v}{\sqrt{\tau_{\text{life}}^2 + \tau_v^2}}, \tag{1}$$

where $\tau_v = w_\phi / v_\phi, w_\phi$ is the poloidal correlation length and v_ϕ is the poloidal flow velocity. From this formula it is clear that we have two distinct limiting cases, which are given below.

(i) $\tau_{\text{life}} \gg \tau_v$: For a fixed w_ϕ spatial correlation length the correlation time τ_{corr} depends mainly on v_ϕ; $\tau_{\text{corr}} \approx w_\phi / v_\phi$. In

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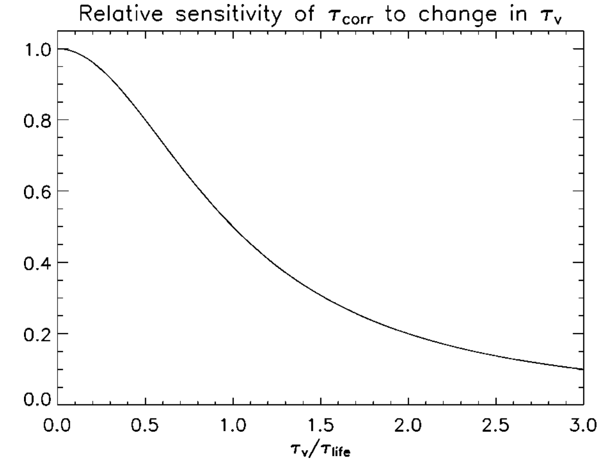

FIG. 1. Ratio of τ_{corr} modulation to τ_v modulation $(d\tau_{\mathrm{corr}}/\tau_{\mathrm{corr}})/(d\tau_v/\tau_v)$ as a function of $\tau_v/\tau_{\mathrm{life}}$.

this case from the correlation time we can follow the time evolution of the $v_\phi(t)$ flow velocity, therefore we can calculate the spectrum of velocity fluctuations.

(ii) $\tau_{\mathrm{life}} \ll \tau_v$: In this case the correlation time gives information about the eddy lifetime and the flow velocity cannot be deduced. It has to be noted that in this case the velocity determination from the cross-correlation function also becomes difficult due to the small shift of the maximum relative to the width.

Of course the situation can be more complicated when the eddy lifetime is in the same order of magnitude as τ_v. In this case we have to do poloidally resolving[14] or two-dimensional (2D) (radial-poloidal) measurements[15,16] in order to distinguish the effect of velocity from the eddy turnover time. However, if τ_{life} does not depend on the flow velocity, then the modulation of the τ_{corr} correlation time will reflect the modulation of the flow velocity, albeit with a sensitivity depending on the $\tau_v/\tau_{\mathrm{life}}$ ratio. This is shown in Fig. 1, where $(d\tau_{\mathrm{corr}}/\tau_{\mathrm{corr}})/(d\tau_v/\tau_v)$ is plotted as a function of $\tau_v/\tau_{\mathrm{life}}$ on the basis of Eq. (1).

The method presented in this paper assumes $\tau_{\mathrm{life}} \gg \tau_v$. We intend to derive the flow velocity fluctuations from modulation of the τ_{corr} correlation time determined from short time signals as is shown in Fig. 2. There are clearly cases when the $\tau_{\mathrm{life}} \gg \tau_v$ relation is supported by experimental results[13] but, of course, it has to be verified case by case.

To assess the minimum length of the time interval needed for the determination of the correlation function (and thus the temporal resolution of the method) we need to calculate the statistical scatter of the correlation function. This scatter arises from two sources: (i) In a short time interval we observe only a limited number of turbulence events and random overlapping between different events produces a statistical scatter of the calculated correlation values, (ii) the measured signal often includes some kind of detector noise, e.g., due to the limited number of photons detected in BES or electron cyclotron emission (ECE) measurements. In the following sections we calculate the relative scatter of the autocorrelation function due to these effects.

III. RELATIVE SCATTER OF THE AUTOCORRELATION FUNCTION DUE TO FINITE NUMBER OF EVENTS

We do the calculations in two steps. In the first part we derive the tendencies without any assumption on the temporal shape of the fluctuation events. In the second part we calculate the detailed expression for the case of uniform Gaussian pulses.

A. General case

In our calculation we assume that we have a measured time signal which consists of randomly distributed, limited temporal length events, including overlapping as well. Each event has a time evolution and it is described by a set of random variables (with a given but otherwise arbitrary distribution function) such as the amplitude, the lifetime, the time center, etc. As we would like to calculate statistical properties we have to define different statistical averages or expectation values. We will use two different notations. The first one is for the time averages:

$$\bar{S}(t) = \frac{1}{\Delta T}\int_0^{\Delta T} S(t)dt \underset{\Delta T \gg w_t}{\underbrace{\longrightarrow}} \frac{1}{\Delta T}\int_{-\infty}^{+\infty} S(t)dt,$$

where $S(t)$ is the measured time signal and w_t is the expectation value of the event lifetime. The second notation is used for averages over random variables describing all the events which constitute the $S(t)$ signal:

$$\langle S(t;\mu)\rangle_\mu = \int \mathcal{P}_\mu S(t;\mu)d\mu, \qquad (2)$$

where $\mu=\{\xi_1,\xi_2,\ldots,\xi_i,\ldots,\xi_n\}$ represents a set a random variables with a set of distributions $\mathcal{P}_\mu = \{P(\xi_1),P(\xi_2),\ldots,P(\xi_i),\ldots,P(\xi_n)\}$. This way the formal definition written above involves multiple integrals over each of the random variables.

If we have a time signal $S(t)$ the autocovariance function (autocorrelation without normalization) can be calculated as

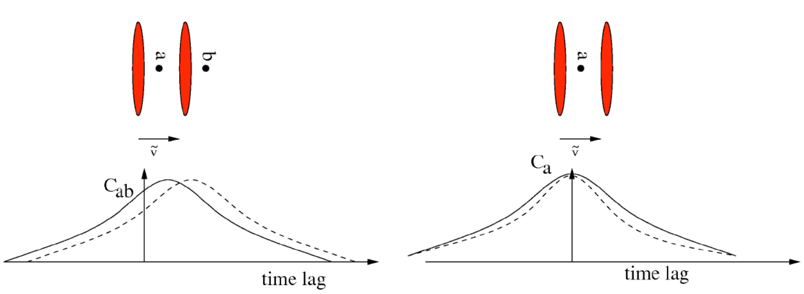

FIG. 2. Sketch of two-point (left) and one-point (right) correlation measurements for velocity fluctuations. In the two-point measurement the maximum place of the correlation function shifts due to the velocity change. In the one-point measurement the width of the autocorrelation function changes.

$$C_a(\tau) = \overline{[S(t) - S][S(t+\tau) - S]}. \quad (3)$$

Our signal is a sum of N_s independent events:

$$S(t) = \sum_{j=1}^{N_s} s_j(t;\mu_j), \quad (4)$$

where $\mu_j = \{A_j, t_{0j}, w_{tj}, \ldots\}$ denotes the set of random parameters describing the jth event with amplitude A_j, time center t_{0j}, lifetime w_{tj}, etc.

The variance of the autocorrelation function is given by

$$\sigma^2 = \langle C_a^2 \rangle - \langle C_a \rangle^2. \quad (5)$$

Using definition (3) we have

$$\sigma^2 = \langle \overline{S(t)S(t+\tau)}^2 \rangle - \langle \overline{S(t)S(t+\tau)} \rangle^2 + \langle \overline{S^4} \rangle - \langle \overline{S^2} \rangle^2 = \sigma_1^2 + \sigma_2^2. \quad (6)$$

We will use the following abbreviations:

$$\sigma_1^2 = \langle C_0^2 \rangle - \langle C_0 \rangle^2, \qquad \sigma_2^2 = \langle \overline{S^4} \rangle - \langle \overline{S^2} \rangle^2, \quad (7)$$

where C_0 denotes the covariance function without average subtraction. Now we calculate these terms separately,

$$\begin{aligned}\langle C_0 \rangle &= \left\langle \frac{1}{\Delta T} \int \sum_{j=1}^{N_s} s_j(t;\mu_j) \sum_{k=1}^{N_s} s_k(t+\tau;\mu_k) dt \right\rangle_\mu \\ &= \frac{1}{\Delta T} \sum_{j=1}^{N_s} \left\langle \int s_j(t;\mu_j) s_j(t+\tau;\mu_j) dt \right\rangle_{\mu_j} + \frac{1}{\Delta T} \sum_{l \neq m} \left\langle \int s_l(t;\mu_l) s_m(t+\tau;\mu_m) dt \right\rangle_{\mu_l,\mu_m} \\ &= \frac{N_s}{\Delta T} \langle \tilde{c}_j(\tau;\mu_j) \rangle_{\mu_j} + \frac{N_s(N_s-1)}{\Delta T} \langle \tilde{c}_{lm}(\tau;\mu_l,\mu_m) \rangle_{\mu_l,\mu_m},\end{aligned} \quad (8)$$

where

$$\tilde{c}_j(\tau;\mu_j) = \int s_j(t;\mu_j) s_j(t+\tau;\mu_j) dt \quad (9)$$

is the autocovariance of one event and

$$\tilde{c}_{lm}(\tau;\mu_l,\mu_m) = \int s_l(t;\mu_j) s_m(t+\tau;\mu_j) dt \quad (10)$$

is the pair covariance function of two different events. Here $\tilde{c}_j$ and $\tilde{c}_{lm}$ are the usual covariance functions multiplied with ΔT and $\tilde{c}_{lm}$ determines the variance of the autocorrelation function. We suppose that different events are statistically independent and the set of $\mathcal{P}_{\mu j}$ probability distributions is identical for all N_s events, i.e., we suppose that events are statistically identical. In this case it can be shown that

$$\langle \tilde{c}_{lm}(\tau;\mu_l,\mu_m) \rangle_{\mu_l,\mu_m} = \Delta T \langle \overline{s_j} \rangle_{\mu_j}^2. \quad (11)$$

Here $\overline{s_j}$ is the time average of event j.

After some calculations, we arrive at the following expression for σ_1 and σ_2:

$$\begin{aligned}\sigma_1^2 &= \frac{N_s}{(\Delta T)^2} \left[\langle \tilde{c}_j^2 \rangle_{\mu_j} - \langle \tilde{c}_j \rangle_{\mu_j}^2 \right] + \frac{N_s(N_s-1)}{(\Delta T)^2} \left[\langle \tilde{c}_{lm}^2 \rangle_{\mu_l,\mu_m} - \langle \tilde{c}_{lm} \rangle_{\mu_l,\mu_m}^2 \right], \\ \sigma_2^2 &= \left[N_s \langle \overline{s_j^4} \rangle_{\mu_j} + 2N_s(N_s-1) \langle \overline{s_j^2} \rangle_{\mu_j}^2 - N_s(N_s-1) \langle \overline{s_j} \rangle_{\mu_j}^4 - N_s^2 \langle \overline{s_j^2} \rangle_{\mu_j}^2 \right].\end{aligned} \quad (12)$$

In order to obtain $\sigma/\langle C_a \rangle$, the relative scatter of the autocorrelation function, we need the expectation value of $C_a(\tau)$,

$$\begin{aligned}\langle C_a(\tau) \rangle &= \langle \overline{S(t)S(t+\tau)} \rangle - \langle \overline{S^2} \rangle \\ &= \frac{N_s}{\Delta T} \langle \tilde{c}_j(\tau;\mu_j) \rangle_{\mu_j} + \frac{N_s(N_s-1)}{\Delta T} \langle \tilde{c}_{lm}(\tau;\mu_l,\mu_m) \rangle_{\mu_l,\mu_m} - \frac{N_s^2}{(\Delta T)^2} \langle \tilde{s}_j \rangle_{\mu_j}^2,\end{aligned} \quad (13)$$

where $\tilde{s}_j = \int s_j(t;\mu_j) dt$.

In the limit of $N_s \gg 1$ and taking into account Eq. (11) the last two terms cancel in the above expression.

Using Eqs. (12) and (13) one can directly calculate $\sigma/\langle C_a \rangle$. However, in the general treatment it is rather complicated therefore we focus here on deriving the tendency in ΔT and N_S, and show the detailed result only for identical Gaussian pulses in the following section.

First let us assess the correlation integrals (9) and (10). For $\tau > w$ (where w is the mean lifetime of the events) these vanish. Additionally the $\tilde{c}_{lm}$ pair covariance function will also vanish if $|t_{0l} - t_{0m}| > w$ holds for the t_0 time centers of events l and m. Otherwise it can be approximated as $D(\mathcal{P}_\mu) A^2 w$, where A is the mean amplitude of the events and the constant D carries the information on the probability distributions. As we assume that the t_0 time centers of the events are uniformly distributed over the ΔT time interval, the statistical averaging brings in a $w/\Delta T$ multiplier. Averaging over all other μ parameters of the events will produce a constant factor which depends neither on N_s nor on ΔT. Based on the above at $\tau = 0$ we have

$$\begin{aligned}&\langle \tilde{c}_j \rangle \propto A^2 w, \quad \langle \tilde{c}_j^2 \rangle \propto A^4 w^2, \quad \langle \tilde{c}_{lm}^2 \rangle \propto A^4 w^2 \frac{w}{\Delta T}, \\ &\langle \tilde{c}_{lm} \rangle \propto A^2 w \frac{w}{\Delta T}, \quad \langle \overline{s_j} \rangle \propto A \frac{w}{\Delta T}, \quad \langle \overline{s_j^2} \rangle \propto A^2 \frac{w^2}{(\Delta T)^2}, \\ &\langle \overline{s_j^4} \rangle \propto A^4 \frac{w^4}{(\Delta T)^4}.\end{aligned} \quad (14)$$

Assuming that $N_s \gg 1$, hence keeping the highest order terms in $N_s, \sigma^2/\langle C_a^2(\tau=0) \rangle$ can be expressed using Eqs. (5) and (12)–(14) as

$$\frac{\sigma^2}{\langle C_a^2\rangle} \approx D_1\left(\frac{w}{\Delta T}\right) + D_2\left(\frac{w}{\Delta T}\right)^2. \qquad (15)$$

The constants D_1 and D_2 are determined by details of the $\mathcal{P}_\mu$ probability distributions. If $w/\Delta T \ll 1$ we can keep only the leading term in $w/\Delta T$, thus our final result is

$$\frac{\sigma}{\langle C_a\rangle} \propto \sqrt{\frac{w}{\Delta T}}. \qquad (16)$$

Our result may be surprising as the relative scatter does not depend on the $n_s = N_s/\Delta T$ event rate. One would expect that for higher event rates the statistics should improve. However, in our case the scatter of the correlation function is produced by random overlapping of events; this is proportional to n_S. The mean of the correlation function is also proportional to n_S, therefore the relative scatter will not depend on the event rate. On the other hand, when ΔT is increased at a fixed event rate the N_S number of events increases as well without changing the random coincidence between the events, and as a result the relative scatter decreases. In this sense the number of measured events does improve statistics.

B. Gaussian pulses

In this section we apply this general result to a simple model of Gaussian shaped events with the same amplitude and lifetime, randomly uniformly distributed in ΔT, hence the single random variable is the time center t_{0j} for each pulse:

$$s_j(t;\mu_j) = e^{-(t-t_{0j})^2/2w_t^2}, \quad \mu_j = \{t_{0j}\}.$$

Substituting the Gaussian shape of the $s_j(t)$ events into Eqs. (9) and (10) we get

$$\tilde{s}_j(\mu_j) = \int s_j(\mu_j)dt = \sqrt{2\pi}w_t, \qquad (17)$$

$$\tilde{c}_j(\tau;\mu_j) = \int_{-\infty}^{+\infty} s_j(t;\mu_j)s_j(t+\tau;\mu_j)dt = \sqrt{\pi}w_t e^{-\tau^2/4w_t^2}, \qquad (18)$$

$$\tilde{c}_{lm}(\tau;\mu_l,\mu_m) = \int_{-\infty}^{+\infty} s_l(t;\mu_l)s_m(t+\tau;\mu_m)dt = \sqrt{\pi}w_t e^{-(t_{0l}-t_{0m}+\tau)^2/4w_t^2}. \qquad (19)$$

From this it is clear that

$$\langle \tilde{c}_j^2\rangle_{\mu_j} - \langle \tilde{c}_j\rangle_{\mu_j}^2 = 0,$$

$$\langle \tilde{c}_{lm}\rangle_{\mu_l,\mu_m} = \sqrt{\pi}w_t\frac{1}{(\Delta T)^2}\int\int e^{-(t_{0l}-t_{0m}+\tau)^2/4w_t^2}dt_{0l}dt_{0m} = \frac{2\pi w_t^2}{\Delta T},$$

$$\langle \tilde{c}_{lm}^2\rangle_{\mu_l,\mu_m} = \pi w_t^2\frac{1}{(\Delta T)^2}\int\int e^{-(t_{0l}-t_{0m}+\tau)^2/2w_t^2}dt_{0l}dt_{0m} = \frac{\sqrt{2}\pi^{3/2}w_t^3}{\Delta T}. \qquad (20)$$

In our case σ_2^2 identically equals zero and σ is independent of τ, thus we arrive at

$$\sigma^2 = N_s(N_s-1)\left[\sqrt{2}\pi^{3/2}\left(\frac{w_t}{\Delta T}\right)^3 - 4\pi^2\left(\frac{w_t}{\Delta T}\right)^4\right]. \qquad (21)$$

To obtain the relative scatter we also need $\langle C_a\rangle$ from Eq. (13):

$$\langle C_a\rangle = N_s\sqrt{\pi}\left(\frac{w_t}{\Delta T}\right)e^{-\tau^2/4w_t^2} + N_s(N_s-1)2\pi\left(\frac{w_t}{\Delta T}\right)^2 + N_s^2 2\pi\left(\frac{w_t}{\Delta T}\right)^2. \qquad (22)$$

Finally collecting all terms the relative variance of the correlation function at τ time lag has the form

$$\frac{\sigma}{\langle C_a\rangle} = \frac{\sqrt{(N_s^2-N_s)\left[\sqrt{2}\pi^{3/2}\left(\frac{w_t}{\Delta T}\right)^3 - 4\pi^2\left(\frac{w_t}{\Delta T}\right)^4\right]}}{N_s\left[\sqrt{\pi}\left(\frac{w_t}{\Delta T}\right)e^{-\tau^2/4w_t^2} - 2\pi\left(\frac{w_t}{\Delta T}\right)^2\right]}. \qquad (23)$$

Now we can do two steps of approximation: first, as we have already mentioned above, $w_t/\Delta T \ll 1$, therefore in the numerator we can neglect the fourth order term and in the denominator the second order terms. The second approximation has also been mentioned before: $N_s(N_s-1) \approx N_s^2$. This way we get a very simple expression for the relative variance at $\tau=0$:

$$\frac{\sigma}{\langle C_a\rangle} \approx \frac{N_s\sqrt{\sqrt{2}\pi^{3/2}\left(\frac{w_t}{\Delta T}\right)^3}}{N_s\sqrt{\pi}\left(\frac{w_t}{\Delta T}\right)} = \sqrt{(2\pi)^{1/2}\frac{w_t}{\Delta T}}. \qquad (24)$$

In order to make sure that the analytical calculation and our considerations are correct a simple numerical simulation was performed. Test signals were created by adding Gaussian pulses with random center time and the numerically calculated scatter and relative scatter of the autocorrelation function at $\tau=0$ calculated from an ensemble of 20 correlation values are plotted in Fig. 3.

Additionally to the simple statistical simulation described above a more complete computational analysis was done to illustrate the feasibility of the velocity measurement method. Results are shown in Fig. 4. Identical Gaussian pulses were moved across a single detection channel with a variable flow velocity. The lifetime τ_{life} of these events was much longer than their transit time τ_v over the observation volume and the pulses were always generated before reaching the observation. The flow velocity was modulated sinusoidally with different frequencies and amplitudes. In some cases an additional Gaussian noise was added to the signal simulated with 1 μs time resolution. The autocorrelation

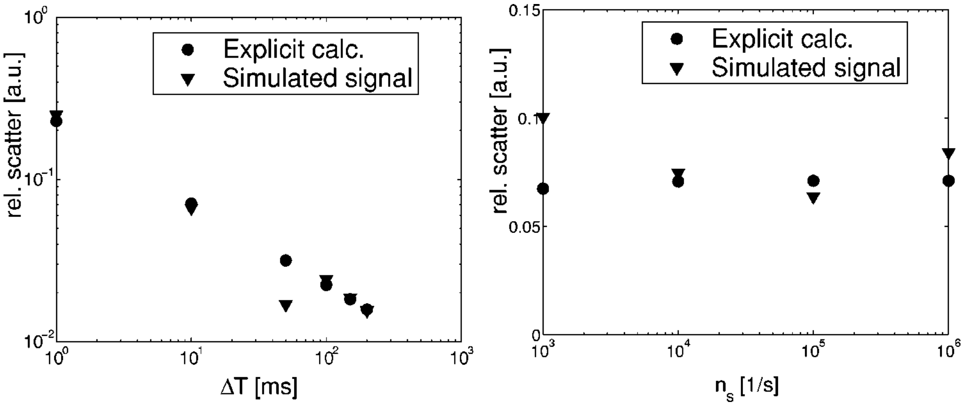

FIG. 3. Comparison between explicit calculation of the relative scatter and a simple numerical model. Constant event rate (left). Constant ΔT (right).

function $C_a(\tau)$ was calculated for short time intervals as plotted in Fig. 4(b). The $\tau_{1/2}$ time lag where the correlation drops to its half was calculated for all of these short time intervals and plotted in Fig. 4(c). The effect of the 100 Hz velocity modulation applied in this simulation is clearly observable in both of these figures. The power spectrum of the $\tau_{1/2}(t)$ signal is shown in Fig. 4(d) for different velocity modulation amplitudes from 0.2 to 0.4. The peak at 100 Hz clearly shows up in all cases. In one of the plotted cases an additional normally distributed random noise was added with a rms amplitude identical to the rms amplitude of the original signal. The resulting power spectrum exhibits a broadband noise and somewhat reduced sensitivity to the 100 Hz modulation, but the peak is clearly distinguishable. This remarkable insensitivity to noise is caused by the $\sqrt{w_t/\Delta T}$ tendency in Eq. (24). The w_t autocorrelation time of the random noise is about 1 μs while for the Gaussian pulses it is about 7 μs [see Fig. 4(c)], thus C_a will be less sensitive to the broadband noise than to event statistical noise.

As a final test the modulation frequency of the flow velocity was raised to 1 kHz and the ΔT time resolution reduced to 100 μs. Although this increases the event statistical noise of the autocorrelation function by nearly a factor of 3, the peak at 1 kHz in the power spectrum in Fig. 4(e) is still apparent.

Finally it should be mentioned that it is a remarkable feature of Eq. (24) that for the calculation of the statistical error of the autocorrelation function only the w_t event correlation time is needed, which can easily be read from the autocorrelation function itself.

IV. IMPLICATION TO PHOTON STATISTICS

In several fusion plasma diagnostic techniques (e.g., BES and ECE) the measured signal contains a significant noise (e.g., detector noise, photon statistical noise). This also causes scatter in the autocorrelation function, albeit with different temporal features. In case of the "event statistical noise," calculated in the preceding section, disturbances consist of w_t wide pulses. In contrast to this the photon statistical noise will cause pulses whose width is determined by the inverse of the amplifier bandwidth. In this section we shall calculate the relative scatter of the autocorrelation function resulting from detector noise. Amplifier noise and photon statistical noise will be treated the same way: a Gaussian white noise can be considered as photon statistical noise in the limit of high photon flux.

Let us assume that the measured signal consists of overlapping photon pulses. The fluctuation of the signal from the plasma modulates the rate of the photons; this way it modulates their statistics as well. In most fusion plasma fluctuation diagnostics we are looking for a small amplitude fluctuation (typically 1%–10%) on top of a slowly varying signal. Ten percent modulation will not change the statistics of the photon noise considerably, therefore we assume that it is constant in time. This enables us to write the detected fluctuation signal as a sum of two signals: the fluctuating plasma signal (or event signal $\tilde{S}_E$) and an additional photon statistical noise signal $\tilde{S}_{\rm ph}$,

$$\tilde{S}(t) = \tilde{S}_E(t) + \tilde{S}_{\rm ph}(t). \tag{25}$$

The two signals are now uncorrelated. Assuming that at least one of the above signals have a zero mean the autocorrelation function can be written as

$$\begin{aligned} C_a(\tau) &= \langle [S_E(t) + S_{\rm ph}(t)][S_E(t+\tau) + S_{\rm ph}(t+\tau)]\rangle \\ &= \langle S_E(t)S_E(t+\tau)\rangle + \langle S_{\rm ph}(t)S_{\rm ph}(t+\tau)\rangle + \langle S_{\rm ph}(t)S_E(t \\ &\quad + \tau)\rangle + \langle S_E(t)S_{\rm ph}(t+\tau)\rangle = C_a^E + C_a^{\rm ph}. \end{aligned} \tag{26}$$

This result shows the additivity of the autocorrelation function. To calculate the squared variance of C_a, we can simply sum the squared variances of the two terms. We have to note that the relative variance of the photon noise can be calculated exactly the same way as in the case of event statistics because the photon noise can be described as a sum of events (photons) with a lifetime determined by the time constant of the amplifier.

Now we wish to calculate the ratio of the variance of the correlation function arising from event statistics and photon noise. Rewriting Eq. (24) we get

$$\sigma_{C_a} = \sqrt{(2\pi)^{1/2}\frac{w_t}{\Delta T}\langle C_a(\tau=0)\rangle} = \sqrt{(2\pi)^{1/2}\frac{w_t}{\Delta T}}A_{\rm rms}^2, \tag{27}$$

where $\langle C_a(\tau=0)\rangle$ was identified as the square of the rms amplitude $A_{\rm rms}$ of the fluctuating signal. For Gaussian pulses

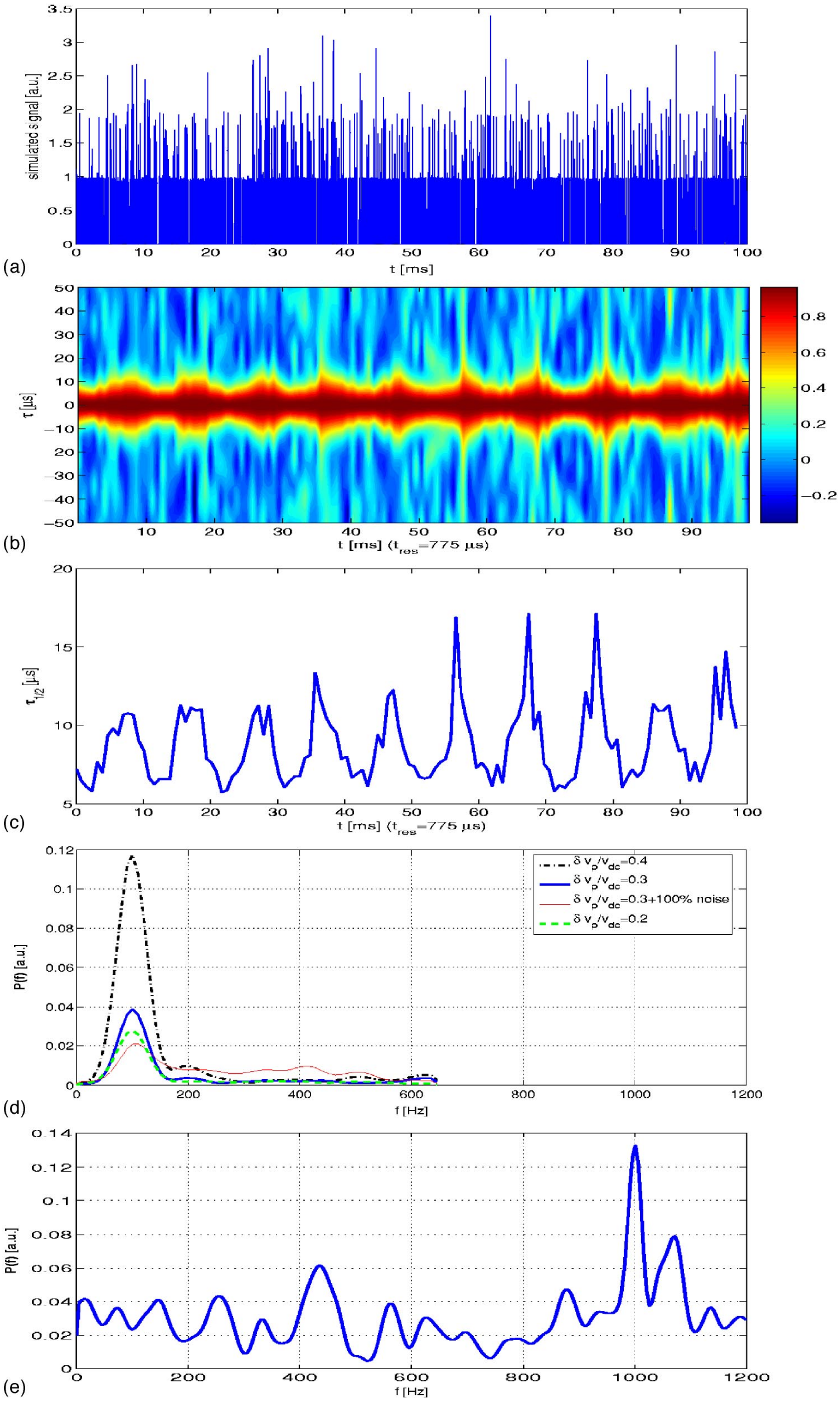

FIG. 4. Results of the full 1+1 dimensional simulation. Modulation frequency of the poloidal flow velocity is 100 Hz for (a)–(d) and 1 kHz for (e). The relative amplitude of the velocity modulation ($\delta v_p/v_{\rm dc}$) is 0.3 except for the cases noted in (d). (a) Simulated signal composed of identical Gaussian pulses. (b) Autocorrelation function vs time. (c) Half width of the autocorrelation function ($\tau_{1/2}$) as a function of time. (d) Power spectrum of the $\tau_{1/2}(t)$ signal for various relative velocity modulation amplitudes and additional random noise for $\delta v_p/v_{dc}$=0.3. (e) Power spectrum of the $\tau_{1/2}(t)$ signal in the case of 1 kHz frequency modulation.

$A_{\rm rms}$ can also be expressed using Eq. (22) from the pulse rate n, the pulse amplitude A, and the pulse width w as $A_{\rm rms}^2 = \sqrt{\pi} wnA^2$. Using these expressions we express the variance due to the event statistics using the rms fluctuation level $A_{E,\rm rms}$ and the variance due to the photon statistics using the photon pulse amplitude $A_{\rm ph}$:

$$\sigma_{C_a}^{E} = \sqrt{(2\pi)^{1/2}\frac{w_t}{\Delta T}A_{E,\rm rms}^2}, \qquad (28)$$

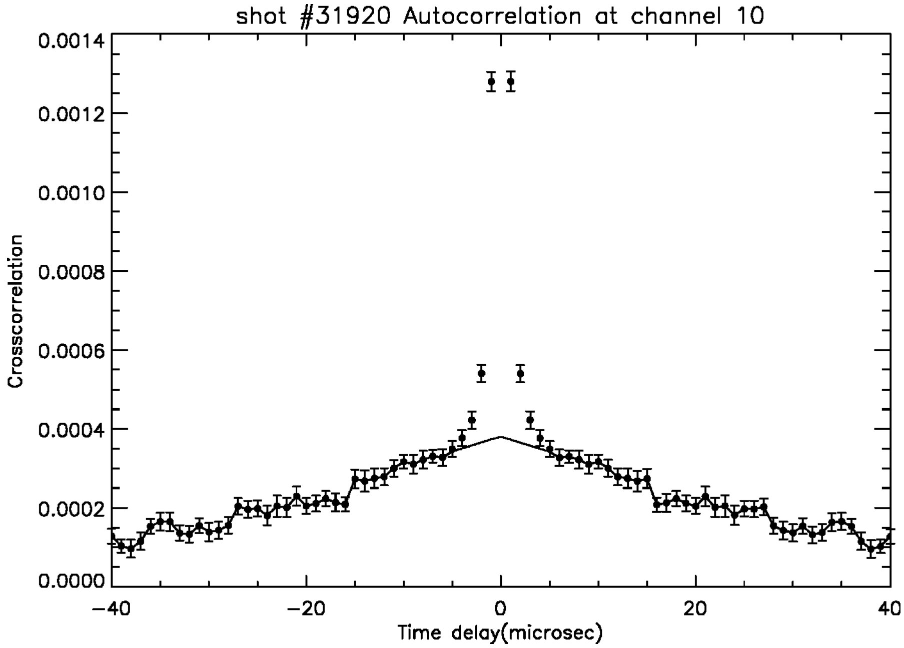

FIG. 5. Extrapolation of the autocorrelation function to separate the photon peak from the correlation function originating from the plasma fluctuations.

$$\sigma_{C_a}^{ph} = \sqrt{(2\pi)^{1/2}\frac{w_{ph}}{\Delta T}\sqrt{\pi}w_{ph}n_{ph}A_{ph}^2}. \quad (29)$$

Here n_{ph} is the photon rate, w_{ph} is the temporal width of the photon pulses (time constant of the amplifier/detector), and A_{ph} is the amplitude of the photon pulses. We can also link these parameters to the dc level of the light signal (A_{dc}) as

$$A_{dc} = \sqrt{\pi}A_{ph}w_{ph}n_{ph}. \quad (30)$$

Using Eq. (30) the ratio of the variance due to photon statistics to the variance due to event statistics is given by

$$\frac{\sigma_{C_a}^{ph}}{\sigma_{C_a}^{E}} = \sqrt{\frac{w_{ph}}{w_t}}\left(\frac{A_{dc}}{A_{E,rms}}\right)^2\frac{1}{\sqrt{\pi}w_{ph}n_{ph}}. \quad (31)$$

We can recognize that the parameters determining the ratio of the two variances are the relative fluctuation amplitude $1/R \equiv (A_{dc}/A_{E,rms})$, the number of photons detected during the time constant of the amplifier $w_{ph}n_{ph}$, and the ratio of the amplifier time constant to the event autocorrelation time. These parameters can all be determined from the signal.

V. ESTIMATION OF PARAMETERS FROM THE EXPERIMENT

As we can see from Eq. (31), in order to obtain this ratio we must determine four quantities: w_{ph} (integration time of the amplifier), w_t (correlation time), R (relative fluctuation amplitude), and n_{ph} (photon rate). Each of these can be directly obtained from the experimental signal.

The easiest way to determine w_{ph} is to analyze a signal measured from a nonfluctuating test radiation source (e.g., light signal for BES, blackbody source for ECE). The width of the peak around $\tau=0$ of the autocorrelation function reveals w_{ph}. The rms fluctuation amplitude in a real experimental signal ($A_{E,rms}$) can be calculated from the square root of the autocorrelation function at $\tau=0$ after correction for the photon noise. If the detector/amplifier cutoff frequency is considerably higher than the frequency of plasma fluctuations, a simple procedure can be used to separate the plasma fluctuation from photon noise in the autocorrelation function.[17] As the effect of photon noise in the autocorrelation function is limited to about $0 \leq \tau < 3w_{ph}$ this method linearly extrapolates the autocorrelation function in this range from $3w_{ph} < \tau < 6w_{ph}$ as shown in Fig. 5. An alternative method would be the separation of the two signal components in the frequency domain.

The extrapolated correlation value at $\tau=0$ approximates $A_{E,rms}^2$. A_{dc} can be calculated as the mean of the signal. The autocorrelation time w_t can be read from the photon peak corrected autocorrelation function.

A. Determination of the photon rate

In this section we present a method to determine the photon rate from measurement. The idea is based on the plot of Fig. 6 which shows the relative rms fluctuation amplitude

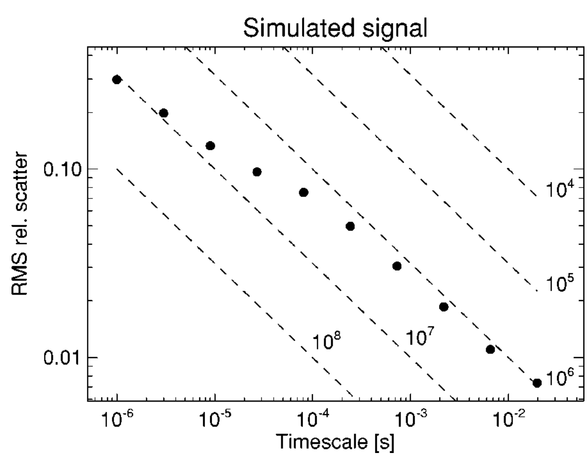

FIG. 6. Determination of the photon rate from a simulated signal (see explanation in the text). The photon rate is 10^7 s^{-1}.

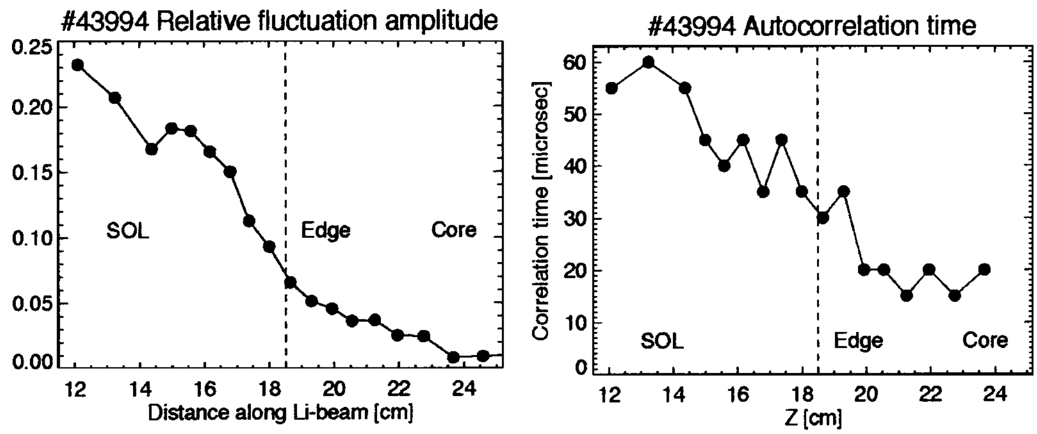

FIG. 7. Experimental determination of the relative rms amplitude (left) and the autocorrelation time (right).

as a function of signal smooth length. If the signal is composed of independent random pulses the relative rms fluctuation amplitude changes with $(n\Delta t)^{-1/2}$ if Δt is considerably longer than the pulse length. The dashed lines represent this dependency in the case of different pulse rates. If at the shortest time scales the photon noise dominates the fluctuation amplitude then the location of the curve relative to dashed lines shows the photon rate. In case the photon amplitude does not dominate one needs to do a measurement with a test radiation source where the dc level of the signal is close to the dc level of the real measurement.

It is worth noting that the plot shown in Fig. 6 is also useful to find characteristic time scales in the signal. Our example shows the presence of about 30 μs wide random pulses in the signal additionally to the photon noise. Unfortunately the n_s event rate cannot be determined from the plot because of a nonfluctuating part of the plasma signal.

VI. SOME CONSIDERATIONS FOR W7-AS Li-BEAM MEASUREMENTS

Using this model we are able to quantify statistical properties of fluctuations measured by Li-beam emission spectroscopy at the Wendelstein 7-AS stellarator[12,17] and to compare fluctuation originating from the scrape-off layer (SOL) and the edge plasma of the machine. Figure 7 shows experimentally measured quantities. We have selected three characteristic locations: (i) SOL located outside the LCFS (last closed flux surface); (ii) edge that is 1–2 cm inside the LCFS; (iii) core that is 5–6 cm inside the LCFS. The data are collected in Table I.

As the table shows in the SOL and edge regions the event statistics dominate the scatter of the correlation function. A few centimeters deeper in the plasma the situation dramatically changes due to the drop in the relative fluctuation amplitude (see Fig. 7). Equation (31) contains the relative fluctuation amplitude squared, therefore this term dominates the expression.

Before attempting an analysis of the flow velocity modulations it is necessary to determine the relation between τ_v and τ_{life}. This can be done by two-dimensional (poloidal-radial) BES.[15,16] This can resolve both the w_ϕ poloidal size and the flow velocity of the structures, but its time resolution is rather limited. From this measurement we see[15] that in the SOL τ_{life} determines the autocorrelation time, while in the edge plasma the effect of the poloidal flow is clearly visible. Although τ_v and τ_{life} appear to be close to each other this only affects the sensitivity of the method (see Fig. 1) and we can attempt to detect modulations in the flow velocity in the edge plasma.

As we would like to detect modulations in the poloidal flow velocity by means of autocorrelation function, it is useful to estimate the uncertainty of the time lag at the half of the maximum correlation which is basically proportional to the uncertainty of the flow velocity determination. This is simple to do using our previous results:

$$Ae^{-\tau_+^2/4w_t^2} = \frac{A}{2} - \sigma,$$

$$\tau_+ = \pm 2w_t\sqrt{\ln\left(\frac{1}{1/2 - \sigma/A}\right)}, \tag{32}$$

$$\Delta\tau_+ := \tau_+ - \tau_{1/2}^{\text{exact}} = 2w_t\left[\sqrt{\ln\left(\frac{1}{1/2 - \sigma_{\text{rel}}}\right)} - \sqrt{\ln 2}\right].$$

TABLE I. Parameters in the SOL, in the plasma edge, and in the outer part of the core plasma.

Location	$w_t(\mu\text{s})$	$w_{\text{ph}}(\mu\text{s})$	$A_{\text{dc}}/A_{E,\text{rms}}$	$n_{\text{ph}}(s^{-1})$	$\sigma^{\text{ph}}_{C_a}/\sigma^{E}_{C_a}$
SOL	50	2	5	10^7	0.14
Edge	20	2	20	10^8	0.35
Core	10	2	50	10^8	3.15

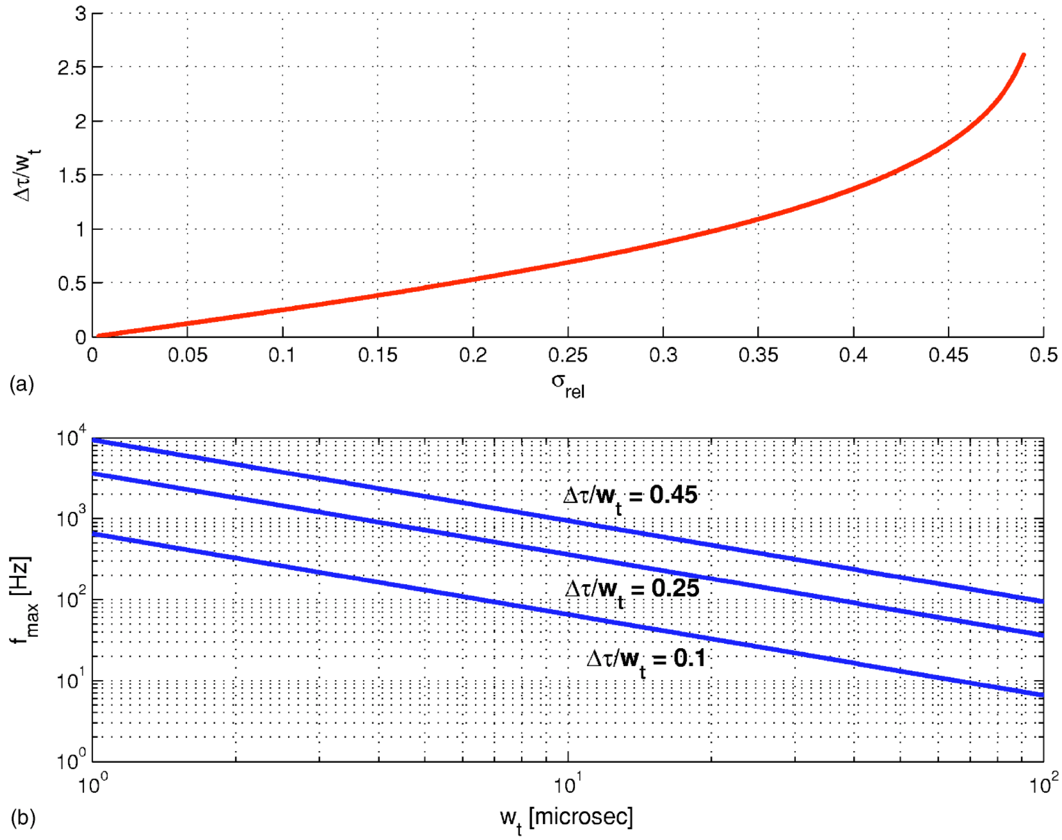

FIG. 8. (a) Uncertainty in determination of the time lag at half maximum (τ_+) of the autocorrelation function with respect to relative scatter of the autocorrelation function [see Eq. (32)] and (b) maximum detectable flow velocity modulation frequency as a function of the mean autocorrelation time w_t at different relative uncertainties of τ_+.

Here σ_{rel} is the relative scatter of the autocorrelation function coming from the statistical noise [see Eq. (24)] and $\tau_{1/2}^{\mathrm{exact}}$ is the time lag at half of the maximum value of the exact autocorrelation function. The ratio of the $\Delta\tau_+(\sigma_{\mathrm{rel}})$ function to w_t is plotted in Fig. 8(a). In case of event statistical noise this expression slightly overestimates the scatter, as the modulation of the maximum of the correlation function is somewhat correlated with modulation at $\tau_{1/2}$.

What we can learn from all these calculations? Let us have an example: assume that we would like to know the modulation in the poloidal flow velocity with an accuracy of $\Delta\tau_+/w_t=25\%$ if the correlation time is about 10 μs. (W7-AS edge conditions.) The question is what has to be the minimum ΔT integration time in order to reach this accuracy. From Fig. 8(a) it is clear that this uncertainty corresponds to 10% relative scatter of the autocorrelation function. From Eq. (24) we can calculate ΔT, which is the minimum sampling time for the flow velocity determination. Consequently the maximum detectable velocity modulation frequency is $f_{\max}=1/(2\Delta T)$. Of course this limiting frequency depends on the needed accuracy in $\Delta\tau_+/w_t$. For longer total measurement times higher scatter can be allowed, but in any case the method breaks down around $\sigma_{\mathrm{rel}}=0.5$. Achievable $f_{\max}$ values are plotted in Fig. 8(b) as a function of the w_t autocorrelation time for various $\Delta\tau_+/w_t$ accuracies. The figure shows that for the Wendelstein 7-AS edge conditions a couple of 100 Hz frequency modulations in the poloidal flow velocity could be detected.

The above considerations indicate that in order to increase the maximum detectable velocity modulation frequency w_t should be decreased. As $w_t=w_\phi/v_\phi$ the ability to detect smaller poloidal structures (better poloidal localization of the measurement) largely increases the detection frequency band. This can be achieved by, e.g., Langmuir probes[10] or by BES with $B_\parallel$ observation.[7]

The above analysis shows that it is possible to detect modulation in the poloidal flow velocity in the edge plasma of the Wendelstein 7-AS stellarator and other fusion devices by means of Li-BES. The statistical method might be easily applicable to probe measurements in the SOL or edge of other devices as well. Such investigations are underway and will be published in the near future.

VII. CONCLUSION

In this paper we have presented a method for the determination of fluctuations in the flow velocity of fusion plasmas. This method is based on the calculation of the autocorrelation function from short time intervals (ΔT) and characterization of flow velocity by means of half width of the autocorrelation function. If we would like to involve short time intervals in our calculations of the correlation function, the knowledge and behavior of statistical errors coming from the finite number of events is crucial. Starting with a simple model of the signal we derived an analytical formula for the relative scatter of the autocorrelation function. Using this expression to our Wendelstein 7-AS fluctuation measurements by Li-beam emission spectroscopy we found that in the SOL and in the edge plasma the main contribution to the error bars comes from the event statistical noise, but in the core plasma the noise of measurement (photon noise) dominates. In addition we also gave a method to determine the photon rate. Finally we are able to relate the relative scatter of the autocorrelation function and the relative uncertainty of the determination of half width of autocorrelation function which is directly connected with the flow velocity in our approximation. As a final result we determined the maximum measurable flow modulation frequency as a function of the autocorrelation time.

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