Non–linear free streaming in Vlasov plasma

Z. Sedláček

Institute of Plasma Physics, Academy of Sciences of the Czech Republic,
Za Slovankou 3, P.O. Box 17, CZ–182 21 Prague 8, Czech Republic

On the basis of a fully non–linear numerical solution of the Vlasov–Poisson equation we demonstrate that in the Fourier transformed velocity space the free streaming is a non–linear multimode phenomenon. In the transformed space the oscillatory part of the disturbance (plasma oscillations) is uncoupled from its free streaming part (the one that in the linearized treatment escapes into infinity) but the free streaming part is strongly coupled to plasma oscillations. It exercises a complicated movement in the Fourier transformed phase plane accompanied by dispersion.

PACS: 52.35.Fp, 52.40.Db
Key words: plasma oscillations, Vlasov equation

1 Introduction

In our recent papers [1, 2, 3, 4, 5] we drew attention to an alternative approach to the problem of Vlasov plasma oscillations which is based on the Fourier transformed velocity space.

We have shown that, in linear approximation, in the Fourier transformed (FT) velocity space the evolution of a disturbance of a Vlasov plasma equilibrium becomes a scattering process on a scattering centre located around the origin of the FT velocity space. In that space the free streaming of particles manifests itself by a simple unidirectional shift of the Fourier transform of the initial perturbation rather than by the creation of fine–scale structures as observed in the original velocity space. The electrostatic interaction of particles creates an oscillating and radiating structure trapped around the origin of the transform space–the Landau damped plasma oscillations.

In linear approximation the plasma oscillations and the free streaming part of the disturbance are uncoupled and completely separated from each other. It is a remarkable fact that this remains partly true even in the fully non–linear treatment of the same problem. In this paper we demonstrate that the plasma oscillations remain uncoupled from the free streaming but the free streaming is now strongly coupled to plasma oscillations. Numerical solution of the non–linear Vlasov–Poisson equation shows that the part of the disturbance that in linear approximation simply escapes into infinity now exercises a complicated movement in the FT phase plane. The free steaming thus becomes a multimode phenomenon. The simulations in addition reveal that the disturbance slowly disperses, predominantly in the wave number space, and disappears almost completely after some tens of plasma periods.

To explain this behaviour we have constructed a simple model based on the observation that the FT Vlasov equation describing the movement of the escaping disturbance becomes approximately the Liouville equation of the linear oscillator with a Hamiltonian periodically dependent on time. Numerical integration of the corresponding canonical equations (equations of the linear oscillator with a periodic restoring force, equivalent to the Mathieu equation) shows the same complicated trajectory in the FT phase plane as actually observed in simulations. Moreover it shows that the movement is oscillatory with periodic recurrence back to the origin of the FT phase plane. But due to its slow dispersion the disturbance disappears before any recurrence can occur. As an important consequence, the well–known filamentation effect in the original phase plane should also slowly disappear.

2 The Fourier transformed Vlasov–Poisson equation

Consider a spatially homogeneous collisionless plasma of electrons with the equilibrium distribution function $F(v)$ normalized to unit density and with immobile ions forming a neutralizing background. Denoting by e, m and N the electron charge, mass and number density, respectively, we write the Vlasov equation for the perturbation distribution function $f(x,v,t)$ as

$$\frac{\partial f(x,v,t)}{\partial t} + v\frac{\partial f(x,v,t)}{\partial x} + \frac{eN}{m}E(x,t)\frac{\partial F(v)}{\partial v} + \frac{eN}{m}E(x,t)\frac{\partial f(x,v,t)}{\partial v} = 0, \quad (1)$$

the perturbation electrostatic field $E(x,t)$ being given by the Poisson equation

$$\frac{\partial E(x,t)}{\partial x} = 4\pi e \int_{-\infty}^{\infty} f(x,v,t)\, dv. \quad (2)$$

We expand the solution into a Fourier series with respect to the spatial coordinate x (periodic boundary condition in space) and Fourier transform with respect to the velocity coordinate v

$$\varphi_m(q,t) = \frac{1}{L}\int_0^L \int_{-\infty}^{\infty} f(x,v,t)\, \mathrm{e}^{imk_1 x}\mathrm{e}^{iqv} dx\, dv, \quad (3)$$

$$\Phi(q) = \int_{-\infty}^{\infty} F(v)\, \mathrm{e}^{iqv}\, dq, \quad (4)$$

$$E_m(t) = \frac{1}{L}\int_0^L E(x,t)\, \mathrm{e}^{imk_1 x} dx, \quad (5)$$

where $m = 0, \pm 1, \pm 2, \ldots$, L is the spatial period and $k_1 = 2\pi/L$ is the wave number of the first space harmonic. The Poisson equation (2) thus becomes

$$E_m(t) = -\frac{4\pi e}{imk_1}\varphi_m(0,t). \quad (6)$$

With this result the Fourier transformed Vlasov–Poisson equation may be written as (we use the standard dimensionless variables)

$$\begin{aligned} &\frac{\partial \varphi_m(q,t)}{\partial t} - mk_1\frac{\partial \varphi_m(q,t)}{\partial q} + q\Phi(q)\frac{1}{mk_1}\varphi_m(0,t) \\ &\quad + {\sum_{l=-\infty}^{\infty}}' q\varphi_l(q,t)\frac{1}{(m-l)k_1}\varphi_{m-l}(0,t) = 0, \end{aligned} \quad (7)$$

where the prime at the sum sign means that the term with $m - l = 0$ is omitted. In order that $f(x, v, t)$ be real, $\varphi_m(q, t)$ must fulfil a condition

$$\varphi_m(q, t) = \overline{\varphi_{-m}(-q, t)}. \tag{8}$$

3 Numerical analysis of the initial value problem

To solve numerically the Fourier transformed Vlasov–Poisson equation (7) we used the method of lines which proved to be simple, efficient and easy to implement. We introduced a numerical grid on the q–axis, utilized a finite–difference scheme to discretize the q–derivative in and then solved the resulting set of ordinary differential equations in time by some suitable ODE solver. For details see [3, 5].

We assume a Maxwellian equilibrium distribution function $F(v)$ and a Maxwellian initial perturbation $f(x, v, 0)$ and excite both the first harmonics $m = \pm 1$ with equal amplitude. The rest of the 16 harmonics used (plus the zero harmonic) has initially zero amplitude. This corresponds to the total initial distribution function of the usual form

$$F(v) + f(x, v, 0) = \frac{1}{\sqrt{2\pi}} \frac{1}{v_{T\mathrm{eq}}} \, \mathrm{e}^{-\frac{1}{2}\left(\frac{v}{v_{T\mathrm{eq}}}\right)^2} (1 + \varepsilon \cos kx) \tag{9}$$

which is an even function of velocity and coordinate so that its Fourier transform is also even. The reality condition (8) then tells us that the Fourier transform is real and stays real at any time. This property enables us to minimize the number of ODE to be solved. The amplitude of the perturbation ε was chosen 0.4.

The results obtained with the program provide some insight into the mechanism of the non–linear interaction between the modes, especially through visualization of the evolution on the q–axis and in the Fourier transformed phase plane (the plane of the wave number k corresponding to the x–coordinate and of the Fourier transformed variable q corresponding to the velocity v).

If we look into the Fourier transformed phase plane we see at the beginning waves propagating in the direction of advection (caused by the second term in (7)) in the positive k half plane and in the opposite direction in the negative k half plane, as in the linear case. After a while a formation resembling a collection of Van Kampen–Case eigenmodes is generated. The non–linear interaction establishes a highly coherent quasistationary wave pattern permanently flowing in a circular manner around the origin of the Fourier transformed phase plane (fig. 1). This flow, among other things, maintains the slowly oscillating electric field.

This quasistationary wave pattern is obviously close to the Fourier transform of a superposition of two BGK modes as conjectured by Demeio and Zweifel [6] and Buchanan and Dorning [7]

$$\varphi_m(q, t) = \frac{1}{2} \, \mathrm{e}^{i(mk_1 t + q)V} \tilde{\varphi}_m(q) + \frac{1}{2} \, \mathrm{e}^{-i(mk_1 t + q)V} \tilde{\varphi}_m(q). \tag{10}$$

One of these modes propagates with velocity V, the other with velocity $-V$. In order that $\varphi_m(q,t)$ be a real and even function of m and q it is sufficient that $\tilde{\varphi}_m(q)$ be a real and even function of m and q.

In the Fourier transformed phase plane the superposition looks like a collection of monochromatic waves, one for each m, propagating in the direction of the q–axis and modulated by a time independent form factor $\tilde{\varphi}_m(q)$. Each wave has the same wave number V and, if $V = \sigma/k$, its frequency σ_m (and its phase velocity) is proportional to m, $\sigma_m = m\sigma$ and this is what is actually observed.

Apart from the wave field in the Fourier transformed phase plane, we observe there another spectacular phenomenon that evolves on the background of the quasistationary wave pattern and that corresponds to the freely propagating scattered initial perturbation escaping into infinity in the linear case. We see the outgoing initial perturbation propagating in the direction of advection while violently oscillating transversely in the direction of the k–axis across many k–modes and in synchronism with the plasma oscillations. The amplitude of these transverse movements gradually increases and becomes so large that the outgoing perturbations penetrate deep into the neighbouring k–half planes, with the opposite sense of advection, where they are somewhat decelerated to produce a characteristic pattern of pirouettes (fig. 2). This phenomenon is responsible for the well known filamentation effect in the original phase plane (the content of higher m and q Fourier components increases).

4 A simple model of the non–linear free streaming

It is not difficult to explain the rather strange behaviour of the scattered disturbance. Since the scattered disturbance practically does not produce any electric field, as in the linear case, the quasistationary wave pattern is entirely uncoupled from the scattered disturbance. Thus, to describe its movement we can consider $\varphi_m(0,t)$ in eq. (7), i.e., the electric field, as given and of approximately constant amplitude so that we can write approximately

$$\frac{\varphi_m(0,t)}{mk_1} = \lambda D_m \tau(t). \tag{11}$$

Since $\varphi_m(0,t)$ is real and even with respect to m, $m = 0, \pm 1, \pm 2, \ldots$, the sequence D_m is odd with respect to m and decreases with $|m|$. The function $\tau(t)$ describes the time dependence of the electric field which is harmonic with very good approximation, $\tau(t) = \cos\omega_1 t$, where ω_1 is the frequency of the lowest Landau eigenmode. The parameter λ characterizes the amplitude of the electric field. The convolution sum may be regarded as an approximate finite difference formula for the derivative with respect to k of a function $\varphi(k,q,t)$ of a continuous variable k, at grid points $k = mk_1$. Since the Fourier transform of the equilibrium distribution function $\Phi(q)$ decreases very rapidly with q we can neglect the third term in eq. (7) so that the equation for the evolution of the distribution function $\varphi(k,q,t)$ of the scattered disturbance approximately becomes

$$\frac{\partial\varphi(k,q,t)}{\partial t} - k\frac{\partial\varphi(k,q,t)}{\partial q} + \lambda q\tau(t)\frac{\partial\varphi(k,q,t)}{\partial k} = 0. \tag{12}$$

This is the Liouville equation of a linear oscillator with a Hamiltonian periodically dependent upon time (a periodic restoring force)

$$H(k,q,t) = \frac{1}{2}k^2 + \frac{1}{2}\lambda\tau(t)q^2. \tag{13}$$

Numerical integration of the corresponding canonical equations of motion (the equations of characteristics of eq. (12))

$$\dot{k} = \lambda\tau(t)q, \quad \dot{q} = -k \tag{14}$$

produces a trajectory in the FT phase plane which is pictured in fig. 2. It is identical with what is actually observed in simulations. Moreover it shows recurrence back to the origin and than again away from the origin and so on. However, finite difference formulas are known to produce dispersion and damping, the more so if they are so imperfect as the one generated by the electric field. The dispersion and probably also the damping of the scattered disturbance is therefore very strong and takes place predominantly in the k direction as clearly visible in fig. 1. The disturbance thus disperses completely before any recurrence can occur.

References

[1] Z. Sedláček and L. Nocera: J. Plasma Phys. **48** (1992) 367.

[2] Z. Sedláček in *International Conference on Plasma Physics ICPP 1994*, Foz do Iguaçu, Brazil 1994 (Eds. P.H. Sakanaka, M. Tendler), no. 345 of American Institute of Physics Proceedings, AIP Press, Woodbury, New York, 1995, p. 119.

[3] Z. Sedláček, Report IPPCZ–361, Institute of Plasma Physics, Academy of Sciences of the Czech Republic, Prague, 1998.

[4] Z. Sedláček and L. Nocera: Phys. Lett. A **296** (2002) 117.

[5] Z. Sedláček and L. Nocera: Czech. J. Phys. **52 Suppl. D** (2002) 65.

[6] L. Demeio and P. F. Zweifel: Phys. Fluids B **2** (1990) 1252.

[7] M. Buchanan and J. Dorning: Phys. Rev. E **50** (1994) 1465.

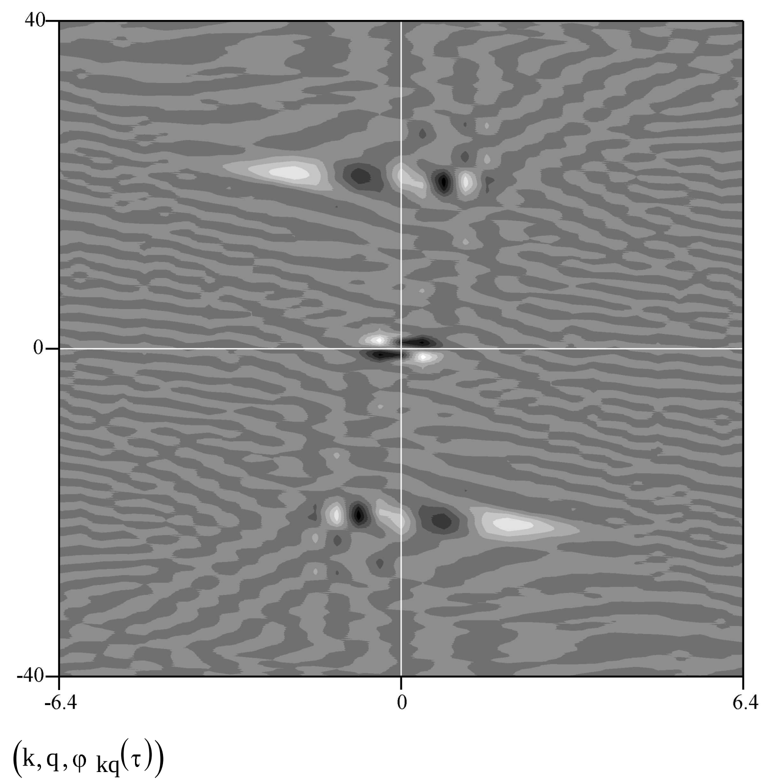

Fig. 1. The fully developed quasistationary wave pattern in the Fourier transformed phase plane ($k_1 = 0.4$, $t = 49.$). The highly dispersed outgoing disturbances just intrude into the neighbouring k half planes.

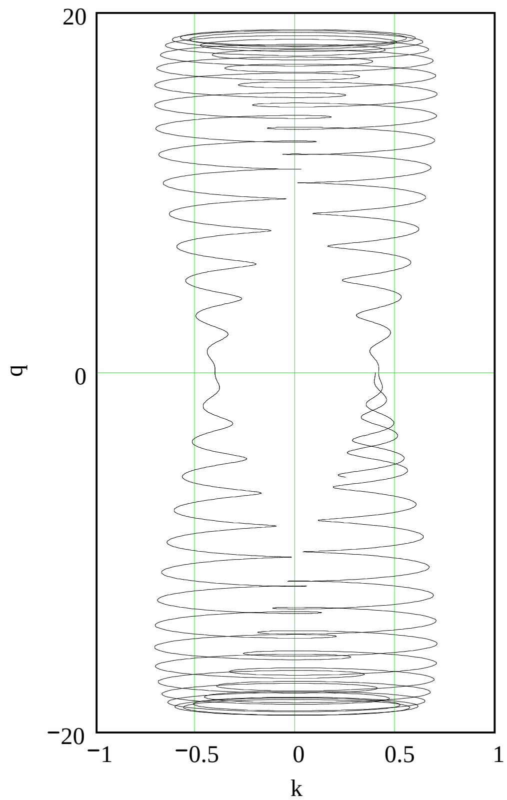

Fig. 2. The trajectory of the scattered disturbance in the FT phase plane as calculated from the simple model of the non–linear free streaming ($k_1 = 0.4$, $\omega_1 = 1.285$, $\lambda = 0.04$). The curve is quasiperiodic, starts at $k = 0.4$, $q = 0$, continues downwards, then turns upwards and returns again to where it started.